ON THE VIBRATIONS OF AN ELASTIC NONHOMOGENEOUS LAYER WITH A CURVILINEAR BOUNDARY, LYING ON AN ELASTIC NONHOMOGENEOUS HALFSPACE

(O KOLEBANII UPRUGOGO NEODNORODNOGO SLOIA S KRIVOLINEINOI GRANITSEI, LEZHASHCHEGO NA UPRUGOM NEODNORODNOM POLUPROSTRANSTVE)

PMM Vol.27, No.1, 1963, pp. 116-125

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(Received October 30, 1962)

The investigation of surface waves is of importance in many questions of theoretical and applied seismology. These problems have been considered in many publications [1,2,3,4] in which the necessary bibliography is to be found. The most complete work in this direction, monograph [5], is devoted to the interference of surface waves and to applications in various domains of seismology.

In this paper we consider the problem of the vibrations of an elastic nonhomogeneous layer with a curvilinear boundary, lying on an elastic nonhomogeneous halfspace. In its precise analytical formulation this problem presents formidable mathematical difficulties. The paper is an attempt to solve, by an approximate method, a suitable reformulation of the exact problem, when the ratio of the wavelength to the thickness of the layer is small, that is to say when the ratio $\lambda/H \ll 1$.

The propagation of waves, under gravity, in a layer of fluid, has been investigated earlier [6], using the same approximation. We shall consider below, using an analogous approximation, the problem of the vibrations of an elastic layer, under the hypothesis that the elastic state depends only on the portion of the nonhomogeneous layer with which it is in contact.

1. Formulation of the problem. Consider an infinite nonhomogeneous elastic layer (medium *1*) lying on an elastic nonhomogeneous half space (medium 2).

We shall suppose that the surface of the layer $y(x)$ is given by $y = -H(x)$, where $H(x) > 0$ is a given function.

Let the layer $0 \geqslant y \geqslant -H(x)$ have the Lamé constants λ_1, μ_1, and the density ρ_1; and the elastic halfspace $y \geqslant 0$ have the Lamé constants λ_2, μ_2, and the density ρ_2. The positive half of the y-axis is supposed to lie inside the halfspace (see Fig. 1).

We shall consider a special type of transverse elastic vibrations propagating in the xy-plane:

$$u_x = 0, \qquad u_y = 0, \qquad u_z = u\,(x,\ y,\ t) \tag{1.1}$$

In this case the equation of elastic vibrations in both media has the form

$$\mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \frac{\partial \mu}{\partial x}\frac{\partial u}{\partial x} = \rho\,\frac{\partial^2 u}{\partial t^2} \tag{1.2}$$

In the absence of external forces on the surface of the layer and in the presence of elastic contact along the entire boundary $y = 0$, the following boundary conditions must hold:

$$\frac{\partial u_1}{\partial x}\frac{dH}{dx} + \frac{\partial u_1}{\partial y} = 0 \quad (y = -H(x)), \quad u_1 = u_2, \quad \mu_1\frac{\partial u_1}{\partial y} = \mu_2\frac{\partial u_2}{\partial y} \quad (y = 0) \tag{1.3}$$

Let us introduce a new dependent variable: $v = \omega y$ where ω is the frequency of the vibrations; equation (1.2) then takes the form

$$\omega_2\,\frac{\partial^2 u}{\partial v^2} + \frac{\partial^2 u}{\partial x^2} + \frac{d\ln\mu}{dx}\frac{\partial u}{\partial x} = \frac{\rho}{\mu}\frac{\partial^2 u}{\partial t^2} \tag{1.4}$$

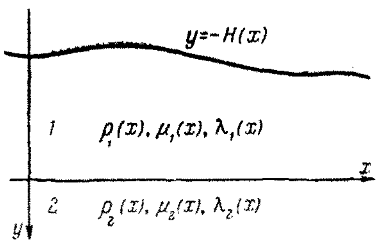

Fig. 1.

The boundary conditions (1.3) then become

$$\omega^2\,\frac{\partial u_1}{\partial v} + \frac{\partial u_1}{\partial x}\frac{dh}{dx} = 0 \quad (v = -h(x))$$

$$u_1 = u_2, \quad \mu_1\frac{\partial u_1}{\partial v} = \mu_2\frac{\partial u_2}{\partial v} \quad (v = 0) \tag{1.5}$$

where we have set $h(x) = \omega H(x)$.

We consider solutions of the equation (1.4) of the surface wave-vibrating layer $0 \geqslant y \geqslant -H(x)$ type, which die out quickly in the halfspace $y \geqslant 0$. These solutions may be represented in the form

$$u_1\,(x,\ v,\ t) = A\,(x,\ v,\ \omega)\cos\,[\alpha\,(x)\,(v + h)]\,e^{i\omega S(x,t)} \tag{1.6}$$

$$u_2\,(x,\ v,\ t) = B\,(x,\ v,\ \omega)e^{-\beta(x)v}e^{i\omega S(x,t)} \qquad (\beta\,(x) > 0) \tag{1.7}$$

where, in view of their physical significance, $\alpha(x)$, $\beta(x)$, $S(x, t)$ are real valued functions; in what follows we shall suppose that $S(x, t) \equiv t - \psi(x)$.

2. Solution of the problem. We shall employ, for the solution of this problem, the asymptotic method, which is developed in [6,7,8]. Let us suppose that for $\omega \gg 1$ the functions $A(x, v, \omega)$ and $B(x, v, \omega)$ may be represented by asymptotic series

$$A(x, v, \omega) \sim A_0(x) + \sum_{n=1}^{\infty} \frac{A_n(x, v)}{(i\omega)^n}, \quad B(x, v, \omega) \sim B_0(x) + \sum_{n=1}^{\infty} \frac{B_n(x, v)}{(i\omega)^n} \quad . \ (2.1)$$

Substituting (1.6) and (1.7) into equation (1.4), taking into account (2.1), and equating to zero the coefficients of the powers of the frequency ω, we obtain an infinite system of differential equations of the second order in the functions A_n and B_n, and also three finite equations for the definition of the functions $\alpha(x)$, $\beta(x)$, $d\psi(x)/dx$ (in view of the corresponding boundary conditions) (2.2)

$$(\nabla\psi)^2 + \alpha^2(x) = \frac{\rho_1(x)}{\mu_1(x)}, \qquad (\nabla\psi)^2 - \beta^2(x) = \frac{\rho_2(x)}{\mu_2(x)}, \qquad \alpha(x) \tan \alpha h = \beta(x)\frac{\mu_2(x)}{\mu_1(x)}$$

$$A_{1vv}\cos\alpha(v+h) - 2\alpha A_{1v}\sin\alpha(v+h) = -\{A_0\cos\alpha(v+h)\nabla^2\psi + \\ + 2\nabla\psi\nabla[A_0\cos\alpha(v+h)] + A_0\cos\alpha(v+h)\nabla\ln\mu_1\nabla\psi\} \qquad (2.3)$$

$$A_{nvv}\cos\alpha(v+h) - 2\alpha A_{nv}\sin\alpha(v+h) + A_{n-1}\cos\alpha(v+h)[\nabla^2\psi + \\ + \nabla\ln\mu_1\nabla\psi] + 2\nabla\psi\nabla[A_{n-1}\cos\alpha(v+h)] = \nabla\ln\mu_1\nabla[A_{n-2}\cos\alpha(v+h)] + \\ + \nabla^2[A_{n-2}\cos\alpha(v+h)] \qquad (n>1) \qquad (2.4)$$

$$[B_{1vv} - 2\beta B_{1v}]e^{-\beta v} = -\{B_0 e^{-\beta v}[\nabla^2\psi + \nabla\ln\mu_2\nabla\psi] + 2\nabla\psi\nabla[B_0 e^{-\beta v}]\} \ (2.5)$$

$$[B_{nvv} - 2\beta B_{nv}]e^{-\beta v} + B_{n-1}e^{-\beta v}[\nabla^2\psi + \nabla\ln\mu_2\nabla\psi] + 2\nabla\psi\nabla[B_{n-1}e^{-\beta v}] = \\ = \nabla\ln\mu_2\nabla[B_{n-2}e^{-\beta v}] + \nabla^2[B_{n-2}e^{-\beta v}] \qquad (n>1) \qquad (2.6)$$

Here, and in what follows, the operator ∇ denotes partial differentiation with respect to x, and it applies only to the first function written immediately after it. Differentiation with respect to v is denoted by a subscript.

Analogously, one obtains the boundary conditions

$$A_{1v} = -A_0\nabla h\nabla\psi, \qquad A_{nv} = -A_{n-1}\nabla h\nabla\psi + \nabla A_{n-2}\nabla h \qquad (v = -h(x)) \quad (2.7)$$

$$A_0\cos\alpha h = B_0, \quad A_n\cos\alpha h = B_n, \quad A_{nv}\cos\alpha h = \frac{\mu_2}{\mu_1}B_{nv} \qquad (v=0) \quad (2.8)$$

$$B_n e^{-\beta v} \to 0 \quad \text{as} \quad v \to \infty \qquad (2.9)$$

From equation (2.2) it follows that the eiconal $\psi(x)$ is defined up to an additive constant, and a sign, which influences only the phase of the vibrations. Subtracting the second equation of (2.2) from the first, we obtain

$$\alpha^2 + \beta^2 = \frac{\rho_1}{\mu_1} - \frac{\rho_2}{\mu_2}$$

From this it follows that surface waves of the type considered may exist only provided that

$$\frac{\rho_1}{\mu_1} - \frac{\rho_2}{\mu_2} > 0, \qquad \text{or} \qquad c_2 > c_1$$

where c_1, c_2 are the speeds of sound in medium *1* and medium 2, respectively. If $\rho_2 \geqslant \rho_1$, then it follows necessarily that $\mu_2 > \mu_1$; that is, the upper layer must lie on a more rigid foundation [9]. The function $\beta(x)$ determines the function $\alpha(x)$ uniquely, by means of the equation

$$\alpha \tan \alpha h = \frac{\mu_2}{\mu_1} \sqrt{\frac{\rho_1}{\mu_1} - \frac{\rho_2}{\mu_2} - \alpha^2}$$

From this equation it follows easily that, corresponding to the conditions

$$h \sqrt{\frac{\rho_1}{\mu_1} - \frac{\rho_2}{\mu_2}} < \pi, \qquad \pi n > h \sqrt{\frac{\rho_1}{\mu_1} - \frac{\rho_2}{\mu_2}} \geqslant \pi (n-1)$$

there exists either one value α^2 (if the first condition holds) or there exist n different values α^2 (if the second condition holds), corresponding to physically different vibrations. Integrating equation (2.3) we obtain

$$A_{nv} = \frac{1}{\mu_1 \cos^2 \alpha (v+h)} \int_{-h(x)}^{v} \Big\{ \cos \alpha (v+h) \nabla [\mu_1 \nabla (A_{n-2} \cos \alpha (v+h))] - \\ - \frac{1}{A_{n-1}} \nabla [A^2_{n-1} \cos^2 \alpha (v+h) \mu_1 \nabla \psi] \Big\} dv + \frac{C_n^* (x)}{\cos^2 \alpha (v+h)} \qquad (2.10)$$

Observing the second boundary condition (2.7), we obtain

$$C_n^* (x) = - A_{n-1} (x, -h(x)) \nabla h \nabla \psi + \nabla A_{n-2} (x, -h(x)) \nabla h$$

After a second integration, from (2.10) we obtain

$$A_n = \frac{1}{\mu_1} \int_0^v \frac{1}{\cos^2 \alpha (v+h)} \Big\{ \int_{-h(x)}^{v} \Big\{ \cos \alpha (v+h) \nabla [\mu_1 \nabla (A_{n-2} \cos \alpha (v+h))] - \\ - \frac{1}{A_{n-1}} \nabla [A^2_{n-1} \cos^2 \alpha (v+h) \mu_1 \nabla \psi] \Big\} dv \Big\} dv + \frac{1}{\alpha} [\nabla A_{n-2} (x, -h(x)) \nabla h - \\ - A_{n-1} (x, -h(x)) \nabla h \nabla \psi] \times [\tan \alpha (v+h) - \tan \alpha h] + C_n (x) \qquad (2.11)$$

Similarly, we may compute $B_n(x, v)$. From equation (2.6) we obtain

$$B_{nv} = \frac{1}{\mu_2} e^{2\beta v} \int\limits_v^\infty \left\{ \frac{1}{B_{n-1}} \nabla \left[B_{n-1}^2 e^{-2\beta v} \mu_2 \nabla \psi \right] - e^{-\beta v} \left[\mu_2 \nabla \left(B_{n-2} e^{-\beta v} \right) \right] \right\} dv + e^{2\beta v} D_n^* (x) \qquad (2.12)$$

According to the condition at infinity, (2.9), $D_n^* = 0$. After a second integration, we obtain for $B_n(x)$ the expression

$$B_n = \frac{1}{\mu_2} \int\limits_0^v e^{2\beta v} \left\{ \int\limits_v^\infty \left\{ \frac{1}{B_{n-1}} \nabla \left[B_{n-1}{}^2 e^{-2\beta v} \mu_2 \nabla \psi \right] - e^{-\beta u} \nabla \left[\mu_2 \nabla \left(B_{n-2} e^{-\beta v} \right) \right] \right\} dv \right\} dv + D_n (x) \qquad (2.13)$$

The recurrence relations (2.11) and (2.13) enable us to compute A_n and B_n for the higher order approximations.

Let us now compute the arbitrary functions $C_n(x)$ and $D_n(x)$. From the second boundary condition (2.8) it follows that

$$C_n (x) \cos \alpha h = D_n (x)$$

Thus the boundary conditions for the nth approximation do not determine the functions C_n and D_n which, however, may be defined in terms of the functions of the next approximation. Indeed, the third condition (2.8) gives an equation for the computation of C_{n-1} and D_{n-1}, which remain undetermined at the $(n - 1)$th approximation. At the $(n - 1)$th step the following functions are defined

$$\overset{\circ}{A}_{n-1} = A_{n-1} - C_{n-1}, \qquad \overset{\circ}{B}_{n-1} = B_{n-1} - D_{n-1} = B_{n-1} - C_{n-1} \cos \alpha h \quad (2.14)$$

Expressing A_{nv} and B_{nv}, for $v = 0$, in terms of A_{n-1}, A_{n-2} and B_{n-1}, B_{n-2}, respectively, by means of (2.14), and substituting these expressions into the third condition (2.8), we obtan

$$- \nabla C_{n-1} \frac{\mu_1 \nabla \psi (\alpha h + \sin \alpha h \cos \alpha h)}{\alpha \cos \alpha h} - C_{n-1} \frac{\nabla \left[(h + \alpha^{-1} \sin \alpha h \cos \alpha h) \mu_1 \nabla \psi \right]}{2 \cos \alpha h} + \frac{1}{\cos \alpha h} \int\limits_{-h(x)}^0 \{ \cos \alpha (v + h) \nabla \left[\mu_1 \nabla \left(A_{n-2} \cos \alpha (v + h) \right) \right] - \qquad (2.15)$$

$$- \frac{1}{\overset{\circ}{A}_{n-1}} \nabla [\overset{\circ}{A}{}_{n-1}^{2} \cos^2 \alpha (v + h) \mu_1 \nabla \psi] \} dv + \frac{\mu_1}{\cos \alpha h} \{ \nabla A_{n-2} (x, - h (x)) \nabla h - \overset{\circ}{A}_{n-1} (x, - h (x)) \nabla h \nabla \psi \} = - \nabla C_{n-1} \frac{1}{\cos \alpha h} \left[\frac{\mu_2{}^2 \nabla \psi \cos^2 \alpha h}{\mu_1 \alpha \tan \alpha h} \right] + C_{n-1} \frac{1}{2 \cos \alpha h} \times$$

$$\times \nabla\left[\frac{\mu_2^2\nabla\psi\cos^2\alpha h}{\mu_1\alpha\tan\alpha h}\right]+\int_0^\infty\left\{\frac{1}{\overset{\circ}{B}_{n-1}}\nabla\left[\overset{\circ 2}{B}_{n-1}e^{-2\beta v}\mu_2\nabla\psi\right]-e^{-\beta v}\nabla\left[\mu_2\nabla\left(B_{n-2}e^{-\beta v}\right)\right]\right\}dv$$

Equation (2.15) is a linear differential equation of the first order which defines the function $C_{n-1}(x)$ up to an arbitrary constant K_{n-1}. In particular, for $n = 1$ we have

$$A_0(x)=C_0(x)=K_0\left[\frac{\mu_1\nabla\psi}{\alpha}\left(\frac{\mu_2^2}{\mu_1^2}\frac{\cos^2\alpha h}{\tan\alpha h}+\sin\alpha h\cos\alpha h+\alpha h\right)\right]^{-1/2} \tag{2.16}$$

Integrating (2.15) we obtain

$$\begin{aligned} C_{n-1}=\frac{A_0}{K_0^3}\Bigg[K_{n-1}+\int_0^x A_0\Bigg\{\int_{-h(x)}^0\Big\{\cos\alpha(v+h)\nabla\Big[\mu_1\nabla\Big[A_{n-2}\cos\alpha(v+h)\Big]\Big]-\\ -\frac{1}{\overset{\circ}{A}_{n-1}}\nabla\Big[\overset{\circ 2}{A}_{n-1}\cos^2\alpha(v+h)\,\mu_1\nabla\psi\Big]\Big\}dv-\\ -\cos\alpha h\int_0^\infty\Big\{\frac{1}{\overset{\circ}{B}_{n-1}}\nabla\Big[\overset{\circ 2}{B}_{n-1}e^{-2\beta v}\mu_1\nabla\psi\Big]-e^{-\beta v}\nabla\left[\mu_2\nabla\left[B_{n-2}e^{-\beta v}\right]\right]\Big\}dv\Bigg\}dx\Bigg] \end{aligned} \tag{2.17}$$

In order to determine $C_n(x)$ one must satisfy the boundary conditions for the $(n + 1)$th approximation.

Employing equations (2.11) and (2.13), we may obtain the first approximations

$$\begin{aligned} A_1=-\frac{1}{2\mu_1A_0}\nabla\left\{\frac{A_0^2\mu_1\nabla\psi}{\alpha}\right\}\left[(v+h)\tan\alpha(v+h)-h\tan\alpha h\right]-\\ -\frac{1}{2\alpha}A_0\nabla\psi\nabla\alpha\left[(v+h)^2-h^2\right]-A_0\nabla\psi\nabla hv+C_1(x) \end{aligned} \tag{2.18}$$

$$B_1=-\frac{1}{2\beta}B_0\nabla\psi\nabla\beta v^2+\frac{1}{2B_0\mu_2}\nabla\left[\frac{B_0^2\mu_2\nabla\psi}{\beta}\right]v+C_1(x)\cos\alpha h \tag{2.19}$$

where A_0 is given by equation (2.16), and $B_0 = A_0 \cos\alpha h$.

3. Remarks on the general character of the approximating functions. The computation of the functions of the second and of higher approximations, and even the function $C_1(x)$, lead to complicated expressions. Let us consider the general nature of the functions $A_n(x,v)$ and $B_n(x, v)$.

Let $Q^{(m)}(v)$ denote a general polynomial of degree m with respect to v, with coefficients which are functions of x. From the preceding it follows that

$$B_0 = Q_0^{(0)}(v), \qquad B_1 = Q_1^{(2)}(v)$$

Suppose that

$$B_{n-1} = Q_{n-1}^{(m)}, \qquad B_{n-2} = Q_{n-2}^{(m-2)}$$

Then, from (2.13) it follows readily that $B_n = Q_n^{(m+2)}$. From this, by induction, it follows that each function B_n is a polynomial of degree $2n$ with respect to v

$$B_n = Q_n^{(2n)} = \sum_{k=0}^{2n} c_{nk}(x)\, v^k \tag{3.1}$$

Analogously, it may be shown that

$$A_n = P_n^{(2n-1)} \tan\alpha(v+h) + R_n^{(2n)} = \tan\alpha(v+h) \sum_{k=0}^{2n-1} a_{nk}(x)\, v^k + \sum_{k=0}^{2n} b_{nk}(x)\, v^k \tag{3.2}$$

where $P_n^{(2n-1)}$ and $R_n^{(2n)}$ are polynomials in v degrees $2n - 1$ and $2n$; and the functions $a_{nk}(x)$, $b_{nk}(x)$ and $c_{nk}(x)$ are rational combinations of the functions

$$A_0(x), \quad \alpha(x), \quad \beta(x), \quad \nabla\psi(x), \quad h(x), \quad \mu_1(x), \quad \mu_2(x), \quad \sin\alpha h, \quad \cos\alpha h$$

and their derivatives.

4. Geometrical optics of propagation of Love waves. The method employed for the expansion in powers of the large parameter ω is essentially the method of short wave approximation, which is valid under the hypotheses that the thickness of the layer $H(x)$ and the radius of curvature R of the surface layer are large with respect to the wavelength λ

$$\lambda = \frac{2\pi c(x)}{\omega} = \frac{2\pi}{\omega}\sqrt{\frac{\mu(x)}{\rho(x)}} \ll H(x), \qquad \lambda \ll R = \frac{[1 + H'^2(x)]^{3/2}}{H''(x)}$$

The equations of the bicharacteristics for the wave process which is described by the equations (1.2) may be written [10]

$$\frac{dx}{dt} = \frac{\mu(x)}{\rho(x)} p_x, \qquad \frac{dy}{dt} = \frac{\mu(x)}{\rho(x)} p_y \qquad \left(p_x = \frac{\partial V(x, y)}{\partial x}, \ p_y = \frac{\partial V(x, y)}{\partial y}\right)$$

$$\frac{dp_x}{dt} = -\frac{1}{2}(p_x^2 + p_y^2)\frac{d}{dx}\frac{\mu}{\rho}, \qquad \frac{dp_y}{dt} = 0 \tag{4.1}$$

where $V(x, y) = t$ is the equation of the wave front. Solving the system (4.1) we obtain the following formulas for the velocity components

$$\frac{dx}{dt} = \pm\frac{\mu(x)}{\rho(x)}\sqrt{\frac{\rho(x)}{\mu(x)} - \frac{1}{a^2}}, \qquad \frac{dy}{dt} = \frac{1}{a}\frac{\mu(x)}{\rho(x)} \tag{4.2}$$

The equations of the ray passing through a point (x_0, y_0) has the form

$$y = y_0 \pm \int_{x_0}^{x} \frac{dx}{\sqrt{a^2/c^2(x) - 1}} \qquad \left(c(x) = \sqrt{\frac{\mu(x)}{\rho(x)}}\right) \tag{4.3}$$

where $c(x)$ is the local speed of sound, a is an arbitrary constant with the physical dimensions of speed, and the condition $c^2(x) \leqslant a^2 \leqslant \infty$ holds.

When $a^2 = c^2(x_0)$, one has $x = x_0$, that is to say, the ray is vertical; while when $a^2 = \infty$ one has $y = y_0$, that is to say, the ray is horizontal. Vertical and horizontal rays do not experience distortion.

Love waves in medium *1* are the result of the superposition of the vibrations which are propagated along the rays (4.3), and the resultant complete internal reflection of these rays on the upper and lower boundary surfaces of the layer. The requirement of complete internal reflection on the boundary $y = 0$ restricts the possible trajectories which produce Love waves in medium *1*. The following condition must be satisfied:

$$\sin\theta \geqslant c_1(x)/c_2(x) \tag{4.4}$$

where the angle θ is the angle of incidence of the ray. From this, as in the case of a medium with constant parameters [11], one obtains again the condition $c_2(x) > c_1(x)$.

Further, suppose that a ray impinges on the plane $y = 0$ at the point $x = x_1$. Then, along the ray the following inequality must hold:

$$\left(\frac{dy}{dx}\right)_{x=x_1} = \cot\theta < \frac{c_2(x_1)}{c_1(x_1)}\sqrt{1 - \frac{c_1^2(x_1)}{c_2^2(x_1)}}$$

From this and from (4.3) we obtain that all rays which are completely reflected internally at the point $(x_1, 0)$, satisfy the equations

$$y = \pm\int_{x_1}^{x} \frac{dx}{\sqrt{a^2/c_1^2(x) - 1}} \qquad \left(\frac{c_1^2(x_1)\,c_2^2(x_1)}{c_2^2(x_1) - c_1^2(x_1)} \leqslant a^2 \leqslant \infty\right) \tag{4.5}$$

where a must satisfy the conditions within the parentheses.

By way of an example, we shall consider the waves generated by an isotropic linear source placed at the point (x_0, y_0).

Consider the vibrations on the surface layer *1* at the point $(x_1, -H(x_1))$. They are produced by a ray which passes directly from the point (x_0, y_0) to the point $(x_1, -H(x_1))$, by a ray which is reflected once from the lower boundary of the layer, by a ray which is reflected twice, once from the upper and once from the lower layer, and so forth (in Fig. 2 several of these rays are dipicted; their distortion corresponds to the case in which $c_1(x)$ decreases monotonically as x increases).

If the function $H(x)$ is given explicitly, then one may calculate numerically, to an arbitrary degree of approximation, the vibration at a point $(x_1, -H(x_1))$, merely by following successively the paths of the first, second, ..., of these rays. Of particular interest are the "direct" ray and the ray which is reflected once (at the point $(x_2, 0)$). The vibrations corresponding to them possess the largest amplitudes, and arrive at the point of observation earlier than the others, produced by the remaining rays. Formulas for these vibrations may be obtained explicitly.

We shall employ a method analogous to that used in [12]. The equation of the ray joining the point (x_0, y_0) to the point $(x_1, -H(x_1))$ has the form (4.3), where the "minus" sign is taken in front of the integral, and the constant "a" is defined by the equation

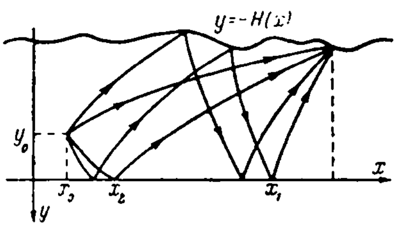

Fig. 2.

$$-H(x_1) = y_0 - \int_{x_0}^{x_1} \frac{dx}{\sqrt{a_2 / c_1^2(x) - 1}} \tag{4.6}$$

Consider the infinitely narrow cone of rays issuing from the point (x_0, y_0) within the angle $d\varphi$. The energy flux in this cone equals

$$Q^2\rho_1(x_1)\, c_1(x_1)\, \omega^2 d\sigma = \frac{P}{2\pi}\, d\varphi \tag{4.7}$$

where $d\sigma$ is the cross-section of the cone at the point, P is the linear strength of the source, and Q is the amplitude at the point $(x_1, -H(x_1))$.

Let us compute $d\sigma/d\varphi$ and substitute it in (4.7). We obtain

$$Q = \omega\left(\frac{P}{2\pi\mu_1(x_1)a}\right)^{1/2}\left\{\sqrt{\left(\frac{a^2}{c_1^2(x_0)}-1\right)\left(\frac{a^2}{c_1^2(x_1)}-1\right)}\int_{x_0}^{x_1}\frac{dx}{c_1^2(x)\,(a^2/c_1^2(x)-1)^{3/2}}\right\}^{-1/2} \tag{4.8}$$

The phase shift is proportional to the frequency

$$\Delta\varphi = \omega\int_{x_0}^{x_1}\frac{a\,dx}{c_1^2(x)\sqrt{a^2/c_1^2(x)-1}} \tag{4.9}$$

The ray which is reflected at the point $(x_2, 0)$ is described, before reflection, by the equation

$$y = y_0 + \int_{x_0}^{x} \frac{dx}{\sqrt{b^2 / c_1^2(x) - 1}} \tag{4.10}$$

and after reflection by the equations

$$y = -\int_{x_2}^{x} \frac{dx}{\sqrt{b^2 / c_1^2(x) - 1}} = -y_0 - \int_{x_0}^{x} \frac{dx}{\sqrt{b^2 / c_1^2(x) - 1}} \tag{4.11}$$

The constants b and x_2 are defined by the equations:

$$y_0 = -\int_{x_0}^{x_2} \frac{dx}{\sqrt{b^2 / c_1^2(x) - 1}}, \qquad H(x_1) = y_0 + \int_{x_0}^{x_1} \frac{dx}{\sqrt{b^2 / c_1^2(x) - 1}} \tag{4.12}$$

Besides, the condition for complete internal reflection must be satisfied

$$\frac{c_1^2(x_2)\, c_2^2(x_2)}{c_2^2(x_2) - c_1^2(x_2)} \leqslant b^2 \tag{4.13}$$

Then the amplitude and the phase shift of the vibrations corresponding to this ray are given by formulas entirely analogous to (4.8) and (4.9), but with a replaced by b.

If the layer *1* has constant thickness, $H(x) = H_0$, then the expressions already obtained may be generalized to an arbitrary number of reflections. Suppose that the ray experiences $2n + 1$ reflections. Then the amplitude and the phase are given by equations (4.8) and (4.9), where the constant $a = a_{2n+1}$ is given by the equation

$$(2n+1)\, H_0 = y_0 + \int_{x_0}^{x_1} \frac{dx}{\sqrt{a_{2n+1}^2 / c_1^2(x) - 1}} \tag{4.14}$$

In the case of an even number, $2n$, of reflections, this equation takes the form

$$(2n+1)\, H_0 = -y_0 + \int_{x_0}^{x_1} \frac{dx}{\sqrt{a_{2n}^2 / c_1^2(x) - 1}} \tag{4.15}$$

Besides, at all points of reflection lying on the upper boundary one must have an inequality of the type of (4.13).

5. Other methods of solution. For the solution of the problem under consideration, as well as for other problems, one may employ the original equation of the problem. In order to do this, let us transform the original equation

$$u_{yy} + u_{xx} + \frac{\mu_x}{\mu} u_x = \frac{\rho}{\mu} u_{tt} \tag{5.1}$$

where subscripts denote partial derivatives. Let us set $x = x(\vartheta)$, where ϑ is a new independent variable, and then (1.2) takes the form (5.2)

$$x'^2 u_{yy} + u_{\vartheta\vartheta} + u_\vartheta\left(\frac{\mu'}{\mu} - \frac{x''}{x'}\right) = x'^2 \frac{\rho}{\mu} u_{tt} \qquad \left(\mu' = \frac{d\mu}{d\vartheta},\ x' = \frac{dx}{d\vartheta},\ x'' = \frac{d^2x}{d\vartheta^2}\right)$$

Further, put $u = \xi^{-1}(\vartheta)\,\Psi(\vartheta, y, t)$ and then (5.2) may be rewritten thus

$$x'^2\Psi_{yy} + \Psi_{\vartheta\vartheta} + \Psi_\vartheta\left(\frac{\mu'}{\mu} - \frac{x''}{x'} - 2\frac{\xi'}{\xi}\right) + \\ + \Psi\left[\left(\frac{\mu'}{\mu} + \frac{x''}{x'}\right)\frac{\xi'}{\xi} + 2\frac{\xi'^2}{\xi^2} - \frac{\xi''}{\xi}\right] = x'^2\frac{\rho}{\mu}\Psi_{tt} \tag{5.3}$$

If one now puts $\Psi = T(t)\varphi(\vartheta, y)$ then we obtain the equation

$$x'^2\varphi_{yy} + \varphi_{\vartheta\vartheta} + \varphi_\vartheta\left(\frac{\mu'}{\mu} - \frac{x''}{x'} - 2\frac{\xi'}{\xi}\right) + \\ + \varphi\left[2\frac{\xi'^2}{\xi^2} - \frac{\xi''}{\xi} - \frac{\xi'}{\xi}\left(\frac{\mu'}{\mu} - \frac{x''}{x'}\right) + x'^2\frac{\rho}{\mu}\omega^2\right] = 0 \tag{5.4}$$

Now, let us put $T = T_0 e^{i\omega t}$, where, without loss of generality, we may suppose that $T_0 = 1$. Let us require further that

$$\frac{\mu'}{\mu} - \frac{x''}{x'} - 2\frac{\xi'}{\xi} = 0 \tag{5.5}$$

$$2\frac{\xi'^2}{\xi^2} - \frac{\xi''}{\xi} - \frac{\xi'}{\xi}\left(\frac{\mu'}{\mu} - \frac{x''}{x'}\right) + \left(\frac{\rho}{\mu} - \frac{\rho_0}{\mu_0}\right)x'^2\omega^2 = 0 \tag{5.6}$$

where ρ_0, μ_0 are certain constant values of ρ and μ. Then equation (5.4) becomes

$$\varphi_{yy} + \frac{1}{x'^2}\varphi_{\vartheta\vartheta} = \frac{\rho_0}{\mu_0}\omega^2\varphi \tag{5.7}$$

Integrating (5.5), we obtain

$$x' = \frac{A\mu}{\omega\xi^2} \qquad (A = \text{const}) \tag{5.8}$$

Substituting the value of the difference $(\mu'/\mu) - (x''/x')$ from (5.5) into (5.6), we obtain

$$\xi'' = \left(\frac{\rho}{\mu} - \frac{\rho_0}{\mu_0}\right)x'^2\omega^2\xi = \left(\frac{\rho}{\mu} - \frac{\rho_0}{\mu_0}\right)\frac{A^2\mu^2}{\xi^3} \tag{5.9}$$

The functions $\mu = \mu(x)$ and $\rho = \rho(x)$ are given functions; returning from the variable ϑ anew to the variable x, equation (5.9) now takes the form

$$\xi'' + \left(\frac{\mu'}{\mu} - 2\frac{\xi'}{\xi}\right)\xi' = \omega^2\xi\left(\frac{\rho}{\mu} - \frac{\rho_0}{\mu_0}\right) \tag{5.10}$$

while equation (5.8) becomes

$$d\vartheta = \frac{\omega\xi^2}{A\mu}\,dx \tag{5.11}$$

and equation (5.7), in turn, is converted into

$$\varphi_{yy} + \varphi_{xx} + F(x)\,\varphi_x = \frac{\rho_0}{\mu_0}\,\omega^2\varphi \qquad \left(F(x) = \frac{\mu'}{\mu} - \frac{2\xi'}{\xi}\right) \tag{5.12}$$

Starting with $\rho = \rho(x)$, $\mu = \mu(x)$, we can obtain $\xi = \xi(x)$ from (5.9), and then obtain $F(x)$ from (5.12). Particular solutions of equation (5.12) may be obtained without difficulty.

Thus we obtain, finally, particular solutions of equation (5.1) of the form

$$u_n = \xi_n^{-1}e^{-i\omega_n t}\,\varphi_n(x, y) \tag{5.13}$$

Knowing these particular solutions, we may construct a general solution of equation (5.1) which satisfies the necessary initial and boundary conditions.

Equation (5.4) may be transformed into a more convenient form. Let

$$x' = 1, \qquad x = \vartheta, \qquad 2\frac{\xi'}{\xi} = \frac{\mu'}{\mu}$$

Then, putting $\xi = A\sqrt{\mu}$, equation (5.4) becomes

$$\varphi_{yy} + \varphi_{xx} + \frac{\omega^2}{c^2}\varphi = 0 \tag{5.14}$$

where

$$\frac{1}{c^2} = \frac{\rho}{\mu} + \frac{1}{2\omega^2}\left(\frac{\mu''}{\mu} - \frac{1}{2}\frac{\mu'^2}{\mu^2}\right) = \Phi(x) \tag{5.15}$$

The general solution of equation (5.14) may now be obtained by the usual methods [13]. Let us consider, by an analogous method, another class of solutions. Let

$$\Psi = b(y)\,\varphi(\vartheta, t) \tag{5.16}$$

then equation (5.2) becomes

$$\varphi_{\vartheta\vartheta} + \varphi_{\vartheta}\left(\frac{\mu'}{\mu} - \frac{x''}{x'} - 2\frac{\xi'}{\xi}\right) + \varphi\left[a^2x'^2 + 2\frac{\xi'^2}{\xi^2} - \frac{\xi''}{\xi} - \frac{\xi'}{\xi}\left(\frac{\mu'}{\mu} - \frac{x''}{x'}\right)\right] =$$
$$= \frac{\rho}{\mu}x'^2\varphi_{tt} \qquad \left(a^2 = \frac{b''}{b}\right) \tag{5.17}$$

Letting

$$\frac{\mu'}{\mu} - \frac{x''}{x'} - 2\frac{\xi'}{\xi} = 0, \qquad a^2x'^2 + 2\frac{\xi'^2}{\xi^2} - \frac{\xi'}{\xi}\left(\frac{\mu'}{\mu} - \frac{x''}{x'}\right) = 0 \tag{5.18}$$

one then has

$$\frac{A\mu}{a\xi^2} = x', \qquad \xi''\xi^3 = A^2\mu^2 \tag{5.19}$$

Returning again to the variable x, we must have

$$\xi'' - \xi'\left(\frac{\mu'}{\mu} - 2\frac{\xi'}{\xi}\right) = a^2\xi, \qquad \vartheta' = \frac{a\xi^2}{A\mu}$$

Given $\mu = \mu(x)$, we shall first find $\xi = \xi(x)$ and then $\vartheta = \vartheta(x)$. Then equation (5.17) becomes

$$\varphi_{\vartheta\vartheta} = \frac{1}{c^2}\varphi_{tt} \qquad \left(\frac{1}{c^2} = \frac{\rho}{\mu}x'^2\right) \tag{5.20}$$

which may be solved easily. Let us note that, from a knowledge of $\rho = \rho(x)$, $\mu = \mu(x)$, and $\vartheta = \vartheta(x)$ one may easily determine $\rho = \rho(\vartheta)$ and $\mu = \mu(\vartheta)$.

For particular approximations $\rho(x)$ and $\mu(x)$, the solution of equations (5.14) and (5.20) may be simplified. For example, if one puts $\mu = (ax + b)^2$ and introduces the new function

$$R = (ax + b)\,u = \frac{\mu'}{2a}u \tag{5.21}$$

then equation (5.1) becomes

$$R_{yy} + R_{xx} = \frac{1}{c^2}R_{tt} \qquad \left(\frac{1}{c^2} = \frac{\rho}{\mu} = \frac{\rho(x)}{(ax+b)^2}\right) \tag{5.22}$$

The solution of this equation may be carried out by the methods indicated above, by transforming the boundary conditions to the new variables.

In conclusion, let us note that the characteritics of the fundamental equation, and in particular, of equation (5.1), may be easily obtained if one puts

$$u = A(x, y, t)\,e^{i\omega f(x, y, t)} \tag{5.23}$$

Then, inserting (5.23) into equation (5.1) and supposing that $\omega \to \infty$, we arrive at

$$f_x^2 + f_y^2 = \frac{\rho}{\mu} f_t^2 \tag{5.24}$$

which is the equation of the wave fronts.

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Translated by J.B.D.